Stochastic process

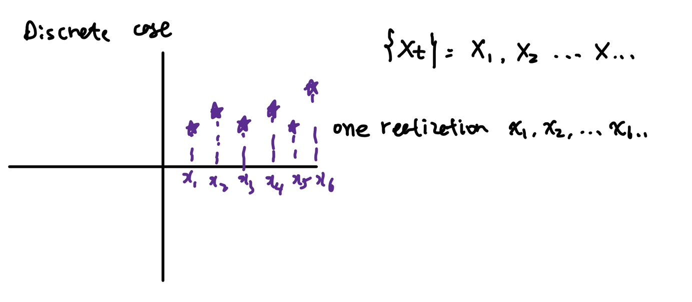
Tuesday, June 11, 2024 21:36

a collection of R.V.s indexed by time.

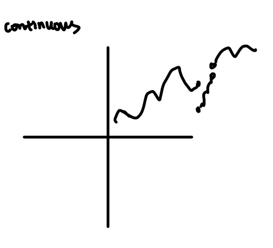
Discrete case

$\{X_t\} = X_1, X_2 \dots X \dots$

one realization $x_1, x_2, \dots x_t \dots$

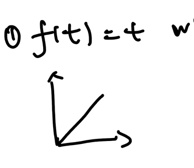


continuous

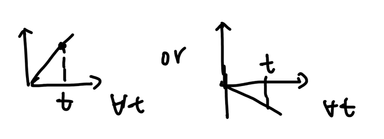


prob. distribution over a space of paths

① $f(t) = t$ with $pr = 1$

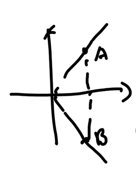


② $f(t) = t$ with $pr = \frac{1}{2}$ $\forall t$, $f(t) = -t$; $\forall t$



at time t, $f(t) = \begin{cases} t & pr = \frac{1}{2} \\ -t & pr = \frac{1}{2} \end{cases}$

③ $f(t) = \begin{cases} t & pr = \frac{1}{2} \\ -t & pr = \frac{1}{2} \end{cases}$ ☆



either A or B, but don't know more than that

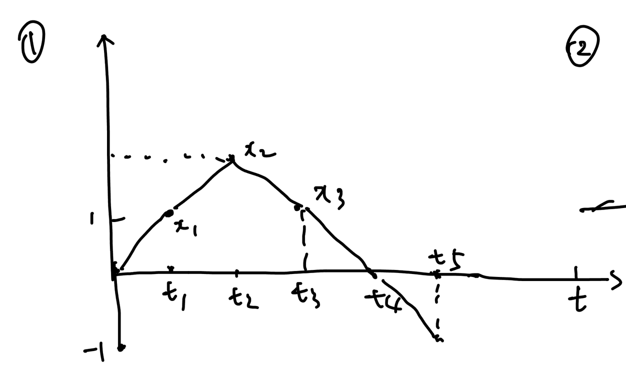
sample random walk

$Y_i = \begin{cases} 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{cases}$

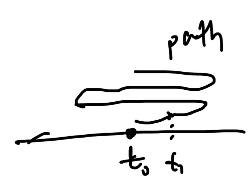
For each time t, $X_i = \sum_{i=1}^{t} Y_i$ $\quad X_0 = 0$

$X_0, X_1, X_2 \dots$ is a one-dimensional simple random walk

①



②



CLT: $\sqrt{n}(\bar{X} - u) \sim N(0, \sigma^2)$

$\therefore X_t = \sum_{i=1}^{t} Y_i$

$\frac{1}{t} X_t = \frac{1}{t} \sum_{i=1}^{t} Y_i$

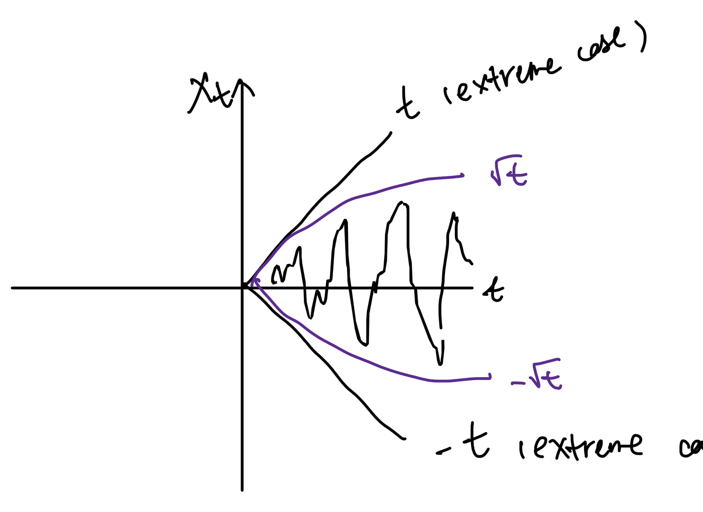
$\sqrt{t}\left(\frac{1}{t} X_t - u\right) \sim N(0, \sigma^2)$

wher u : $E(Y_i) = 0$

σ^2 : $var(Y_i) = 1$

$\Rightarrow \frac{1}{\sqrt{t}} X_t \sim N(0, 1)$

$X_t \sim N(0, t)$



Prop. ① $\mathbb{E} X_k = 0$

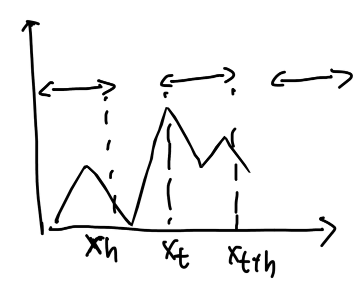
② indep. increment

$0 = t_0 \le t_1 \le \dots \le t_k$

Then $X_{t_{i+1}} - X_{t_i}$ are mutually indep

③ stationary $h \ge 1$, $t \ge 0$

the dist of $X_{t+h} - X_t$ is the same as the dist of X_h



Markov Chain

stochastic processes whose effect of the past on the future is summarized only by current time

simple random walk is a markov chain.

→ what happens next only depends on how high the current value is.

Discrete time stochastic process $X_0, X_1, X_2 \dots$ is a markov chain if

$$P(X_{t+1} = S \mid X_0, X_1, \dots X_t) = P(X_{t+1} = S \mid X_t) \quad \forall t, \forall S.$$

Random walk:

$$P(X_{t+1} = S \mid X_0 \dots X_t) = \begin{cases} \frac{1}{2} & \text{if } S = X_t + 1 \text{ or } S = X_t - 1 \\ 0, & o/w \end{cases}$$

transition matrix:

eigen vector

perron-Frobenius thm. If transition matrix A has positive entries, then there exist a vector $V = \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_n \end{bmatrix}$ s.t.

$AV = V.$ $\quad (\lambda = 1)$

V is called stationary distribution.

if start from $P(X = i) = \pi_i$, the next step has the exact the same distribution

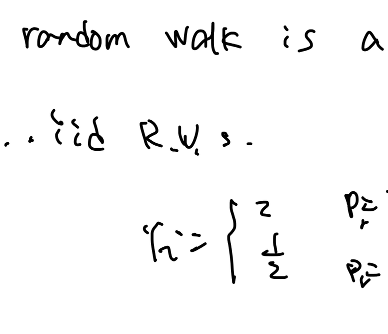
Martingale:

stochastic processes which are fair game.

DEF. stochastic process $\{X_0, X_t, X_2 \dots\}$ is a martingale if

$$X_t = \mathbb{E}(X_{t+1} \mid \mathcal{F}_t)$$

for all $t \ge 0$, $\mathcal{F}_t = \{X_0, X_1, \dots X_t\}$.



the expected value of X_{t+1} centered at X_t

① a simple random walk is a martingale

② $Y_1, Y_2 \dots$ iid R.V.s.

$Y_i = \begin{cases} 2 & p = \frac{1}{3} \\ \frac{1}{2} & p = \frac{2}{3} \end{cases}$

Let $X_0 = 1$ $\quad X_k = \prod_{i=1}^{k} Y_i$

X_k is a martingale

$E(X_{k+1} \mid X_k \dots X_0)$

$= E(Y_{k+1} \cdot Y_k \cdot Y_{k-1} \dots Y_1)$

$= E(Y_{k+1}) \cdot \underbrace{\prod_{i=1}^{k} Y_k}$ → given $X_0, X_1 \dots X_k$ known

$= E(Y_{k+1}) \cdot X_k$

$= 1 \cdot X_k.$

$\left\{ \begin{array}{l} \text{Markov chain} \\ \text{martingale} \end{array} \right.$ are two different things. don't confuse the two

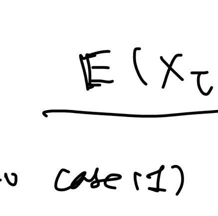
Given a stochastic process, $\{X_0, X_1, \dots\}$. a non-negative integer R.V. τ is called a stopping time if

$\forall$ integer $k \ge 0$, $\tau \le k$ depends only on $X_1, \dots X_k$.

(1) coin toss game: τ be the first time at which balance \$100, or \$-50

↳ stopping time $\quad$ depends on $X_1, \dots X_\tau$

(2) let τ be ~~the time~~ of first peak, not stopping time, depends on $X_{\tau+1}$.



Thm Suppose $X_0, X_1 \dots$ martingale

τ is a stopping time.

$\exists$ constant T, s.t. $\tau \le T$. (τ has bound)

$$\underline{\mathbb{E}(X_\tau) = X_0}$$

apply to case (1)

$E(X_\tau) = 0$

$X_\tau = \begin{cases} 100 & , p \\ -50 & 1-p \end{cases}$

$\therefore 100p + (1-p)(-50) = 0$

$100p + 50p = 50$

$150p = 50$

$p = \frac{1}{3}$